

NOTE

**A SHORT PROOF FOR A PARTITION IDENTITY
OF HWANG AND WEI**

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In Volume 46 of this journal [1] Hwang and Wei prove the following identity:

$$\sum_{p \in P} \prod_{i=1}^m \binom{n_i + 1 - k_i}{k_i} = \sum_{j \geq 0} \binom{j + m - 2}{m - 2} \binom{n + 1 - k - 2j}{k - 2j}$$

for integers n_i with $n_1 + n_2 + \dots + n_m = n$ ($m \geq 2$), where $p = (k_1, \dots, k_m)$ runs through the set P of all partitions $k = k_1 + k_2 + \dots + k_m$ of the nonnegative integer k into m non-negative integers k_i . We give here a very short proof for this identity, using generating functions.

We set

$$A_n(z) = \sum_{k \geq 0} \binom{n + 1 - k}{k} z^k.$$

Then it follows from Riordan [2, p. 154, 2c; note the typos!], that

$$A_n(z) = \mu^{-1} \cdot \left(\frac{-2z}{1-\mu} \right)^{n+2} = \mu^{-1} \cdot \left(\frac{1+\mu}{2} \right)^{n+2} \quad \text{with } \mu = (1+4z)^{1/2}. \quad (1)$$

The identity in question is equivalent to

$$\begin{aligned} \prod_{i=1}^m A_{n_i}(z) &= \sum_{j \geq 0} \binom{j + m - 2}{m - 2} z^{2j} A_{n-4j}(z) \\ &= \sum_{j \geq 0} \binom{-m+1}{j} (-z^2)^j A_{n-4j}(z). \end{aligned}$$

With $n = n_1 + n_2 + \dots + n_m$ and formula (1), we have to show

$$\mu^{-m} \left(\frac{1+\mu}{2} \right)^{n+2m} = \mu^{-1} \left(\frac{1+\mu}{2} \right)^{n+2} \left(1 - \frac{(1-\mu)^4}{16z^2} \right)^{-m+1}.$$

Regarding $4z = -(1+\mu)(1-\mu)$ it follows that

$$1 - \frac{(1-\mu)^4}{16z^2} = 1 - \frac{(1-\mu)^2}{(1+\mu)^2} = \frac{4\mu}{(1+\mu)^2}$$

from which the desired identity is immediate.

References

- [1] F.K. Hwang and V.K. Wei, A partition identity, *Discrete Math.* 46 (1983) 323–326.
- [2] J. Riordan, *Combinatorial Identities* (Wiley, New York, 1968).